

Master stability functions for multiplex networks

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Abstract

Synchronization phenomena are of broad interest across disciplines and increasingly of interest in a multiplex network setting. Here we show how the Master Stability Function, a celebrated framework for analyzing synchronization on a single network, can be extended to multiplex networks with different intra-layer and inter-layer coupling functions. We derive three master stability equations that determine respectively the necessary regions of complete synchronization, intra-layer synchronization and inter-layer synchronization. We calculate these three regions explicitly for the case of a two-layer network of Rössler oscillators coupled via nonidentical intra-layer and inter-layer coupling functions. We find that complete synchronization is stable in the overlapping part of the three regions; intra-layer synchronization is stable in the overlapping part of the complete and the intra-layer synchronized regions; and inter-layer synchronization is stable in the overlapping part of the complete and the inter-layer synchronized regions. Thus, if the inter- or intra-layer coupling function is such that the inter-layer or intra-layer synchronization region is empty, complete synchronization cannot be achieved regardless of the coupling strength. Furthermore, for any given nodal dynamics and network structure, the occurrence of intra-layer and inter-layer synchronization depend mainly on the coupling functions of nodes within a layer and across layers, respectively.

Keywords: Multiplex network; master stability function; intra-layer synchronization; inter-layer synchronization; synchronized region.

Introduction

Synchronization in a network of connected elements is essential to the proper functioning of a wide variety of natural and engineered systems, from brain networks to electric power grids. This has stimulated a large number of investigations into synchronization

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properties of complex networks, with small-world, scale-free and other types of topologies [1–15]. Yet many synchronization phenomena, as in electrical power grids, do not involve a single network in isolation but rely on the global synchronization of a collection of smaller networks. And more generally, beyond single networks, we are now understanding that interactions between networks are increasingly important and that interactions can impact the dynamical processes [16–21]. One paradigm that captures many real-world interdependent networks is that of multiplex networks. Here the same set of nodes interact on different layers, where each layer represents a different edge type, the state of the node on each layer can be distinct, and the connectivity pattern on each layer can be unique [22, 23]. As an example consider the online social system of a set of individuals. They may interact on Twitter or on Facebook or on Linked-in or on some combination of all three, and each layer can have its own connectivity pattern, yet there is typically influence propagated between the layers. Given the need to study dynamical processes on layered complex networks, and the broad applicability of synchronization, here we study synchronization phenomena on multiplex networks, an area that has attracted increasing attention in the past few years [24–33].

One of the most important methods to study network synchronization on isolated networks is the master stability function (MSF) method proposed by Pecora and Carroll [34]. As established via the MSF approach, whether or not a network can achieve synchronization is determined not only by the network structure, but also by the nodal dynamics and the inner coupling function which describes the interactions among the different components of the state vectors of connected nodes [35–37]. Current studies of synchronization phenomena in multiplex networks analyze a multiplex network as a single large composite network with the topology being described by a supra-Laplacian matrix. This requires that the inner coupling function is the same regardless of whether the nodes are linked by an intra-layer or inter-layer edge and the MSF framework can thus be directly applied. The eigenvalues of this supra-Laplacian are then used to analyze the stability of the synchronous state in multiplex networks. For example, Solé-Ribalta et al. [24] investigated the spectral properties of the Laplacian of multiplex networks, and discussed the synchronizability via the eigenratio of the Laplacian matrix. Aguirre et al. [25] studied the impact of the connector node degree on the synchronizability of two star networks with one inter-layer link and showed that connecting the high-degree (low-degree) nodes of each network is the most (least) effective strategy to achieve synchronization. Xu et al. [30] investigated the synchronizability of two-layer networks for

three specific coupling patterns, and determined that there exists an optimal value of the inter-layer coupling strength for maximizing synchronizability of the two-layer networks they analyze. Li et al. [31] investigated the synchronizability of a duplex network composed of two star networks with two inter-layer links by giving an analytical expression containing the largest and the smallest nonzero eigenvalues of the Laplacian matrix, the link weight, as well as the network size.

In 2012, Sorrentino et al. [38, 39] considered an innovative “hypernetwork” model consisting of nodes that interact via multiple types of coupling functions. They were interested in the stability of the completely synchronous state and found three situations where the network topology is such that one can decouple the effects of interaction functions from the structure of the networks and apply the MSF approach. Extremely recently, del Genio et al. extended this analysis to a broader range of scenarios, again using an MSF approach [40], and show how the “hypernetwork” model of [38, 39] is equivalent to a network where nodes have many different interaction types (or “layers” of interaction). Although these works consider that nodes can interact with one another via different coupling functions, they do not capture the richness of phenomena that can occur in multiplex networks. In a multiplex setting each node exists on each layer of the network. So if there are k layers, there are k nodes and each can have a unique state (see Eq. (1) below), enabling phenomena such as intra-layer and inter-layer synchronization. In the hypernetwork setting, although each node can have different types of edges to other nodes, each node occurs only once, so it can be in only one state at any given time, restricting the phenomena to global synchronization.

The works discussed thus far focus on complete synchronization. Yet, as mentioned, in a multiplex network setting, intra-layer and inter-layer synchronization are two other important coherent behaviors. For example, Gambuzza et. al [32] analyzed synchronization of a population of oscillators indirectly coupled through an inhomogeneous medium. The system is formalised in terms of a two-layer network, where the top layer is composed of disconnected oscillators, and the bottom layer consists of oscillators coupled according to a given topology, and each node in the top layer is connected to its counterpart in the bottom layer. By numerical simulations, they have shown the onset of intra-layer synchronization without inter-layer coherence, that is, a state in which the nodes of a layer are synchronized between them without being synchronized with those of the other layer. Shortly afterwards, Sevilla-Escoboza et. al [33] investigated the inter-layer synchronization in a duplex network of identical layers, and showed that each node in a given layer

can synchronize with its replica in the other layer irrespective of whether or not intra-layer synchronization occurs. These findings into specific systems provide useful foundations for elucidating a more fundamental approach to analyzing synchronization phenomena in multiplex networks.

Based on the above motivations, here we develop a Master Stability Function method for multiplex networks. This captures an essential feature of multiplex networks, that the inter-layer coupling function can be distinct from the intra-layer coupling function. Thus, distinct from previous approaches, we can analyze different kinds of coherent behaviors, including complete synchronization, intra-layer synchronization and inter-layer synchronization in multiplex networks. In particular, we derive the master stability equation for a general multiplex network as well as two reduced forms corresponding to only inter-layer or intra-layer interactions. We then show how three different necessary regions for synchronization can be calculated from the MSF of the three master stability equations. Finally we show how to explicitly apply the multiplex MSF by analyzing a specific example of two-layer network of Rössler oscillators with identical intra-layer topological structures and one-to-one inter-layer connections.

Results

A multiplex network model. We consider a multiplex network consisting of M layers each consisting of N nodes. The state of the i -th node in the k -th layer is specified by $\mathbf{x}_i^{(k)} = (x_{i1}^{(k)}, x_{i2}^{(k)}, \dots, x_{im}^{(k)})^\top$, an m -dimensional state vector. The evolution of the full multiplex system can be written as:

$$\dot{\mathbf{x}}_i^{(k)} = f(\mathbf{x}_i^{(k)}) - c \sum_{j=1}^N l_{ij}^{(k)} H \mathbf{x}_j^{(k)} - d \sum_{l=1}^M d_{kl} \Gamma \mathbf{x}_i^{(l)}, \quad i = 1, 2, \dots, N; k, l = 1, 2, \dots, M, \quad (1)$$

where $\dot{\mathbf{x}}_i^{(k)} = f(\mathbf{x}_i^{(k)})$ ($i = 1, 2, \dots, N; k = 1, 2, \dots, M$) describes the isolated dynamics for the i -th node in the k -th layer, and $f(\cdot) : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is a well-defined vector function, $H \in \mathbb{R}^{m \times m}$ and c are the inner coupling function and coupling strength for nodes within each layer, respectively, and $\Gamma \in \mathbb{R}^{m \times m}$ and d are the inner coupling function and coupling strength for nodes across layers, respectively. As can be seen from Eq. (1), the coupling functions between nodes are linear (thus we can also call H and Γ inner coupling matrices). Furthermore, the inner coupling matrix for nodes within one layer, H , is identical for all layers and the inner coupling matrix for nodes across two layers, Γ , is the same for all pairs of layers.

If the i -th node is connected with the j -th node within the k -th layer, $l_{ij}^{(k)} = -1$, otherwise $l_{ij}^{(k)} = 0$, and $l_{ii}^{(k)} = -\sum_{j=1}^N l_{ij}^{(k)}$, for $i, j = 1, 2, \dots, N$ and $k = 1, 2, \dots, M$. Similarly, if a node in the k -th layer is connected with its replica in the l -th layer, $d_{kl} = -1$, otherwise $d_{kl} = 0$, and $d_{kk} = -\sum_{l=1}^M d_{kl}$, for $k, l = 1, 2, \dots, M$.

For simplicity, denote

$$\mathbf{x}^{(k)} = \begin{pmatrix} \mathbf{x}_1^{(k)} \\ \mathbf{x}_2^{(k)} \\ \vdots \\ \mathbf{x}_N^{(k)} \end{pmatrix}, \tilde{f}(\mathbf{x}^{(k)}) = \begin{pmatrix} f(\mathbf{x}_1^{(k)}) \\ f(\mathbf{x}_2^{(k)}) \\ \vdots \\ f(\mathbf{x}_N^{(k)}) \end{pmatrix}, \mathbf{x} = \begin{pmatrix} \mathbf{x}^{(1)} \\ \mathbf{x}^{(2)} \\ \vdots \\ \mathbf{x}^{(M)} \end{pmatrix}, F(\mathbf{x}) = \begin{pmatrix} \tilde{f}(\mathbf{x}^{(1)}) \\ \tilde{f}(\mathbf{x}^{(2)}) \\ \vdots \\ \tilde{f}(\mathbf{x}^{(M)}) \end{pmatrix},$$

then the evolution of the multiplex network (Eq. 1) can be rewritten as

$$\dot{\mathbf{x}} = F(\mathbf{x}) - c(\mathcal{L}^L \otimes H)\mathbf{x} - d(\mathcal{L}^I \otimes \Gamma)\mathbf{x}, \quad (2)$$

where \mathcal{L}^L stands for the supra-Laplacian of intra-layer connections and \mathcal{L}^I for the supra-

Laplacian of inter-layer connections. In detail, $\mathcal{L}^L = \bigoplus_{l=1}^M L^{(k)} = \begin{pmatrix} L^{(1)} & & & \\ & L^{(2)} & & \\ & & \ddots & \\ & & & L^{(M)} \end{pmatrix}$

and $\mathcal{L}^I = L^I \otimes I_N$. Here, \otimes is the Kronecker product, $L^{(k)} = (l_{ij}^{(k)})_{N \times N}$ is the Laplacian matrix of nodes within the k -th layer, and $L^I = (d_{kl})_{M \times M}$ represents the inter-layer Laplacian matrix. More details about supra-Laplacians and multiplex network models can be found in Refs. [18, 24, 26, 30, 31] and references therein.

A master stability function framework for multiplex networks. The master stability function method [34] is one of the most important methods to study stability of synchronized coupled identical systems. It simplifies a large-scale networked system to a node-size system via diagonalization and decoupling, as long as the inner coupling functions for all node pairs are identical. Thus, determining whether a network can reach synchronization can be turned into determining whether all the network characteristic modes fall into the corresponding synchronized regions. In the following, we will establish a master stability framework for multiplex networks with nonidentical inter-layer and intra-layer inner coupling functions.

According to the idea of the master stability framework [34], to investigate network synchronization, we can linearize the dynamical equation (2) at $\mathbf{1}_M \otimes \mathbf{1}_N \otimes \mathbf{s}$, where \mathbf{s} is a synchronous state of the network satisfying $\dot{\mathbf{s}} = f(\mathbf{s})$ and $\mathbf{1}_M$ denotes an M -dimensional vector with all entries being 1. We thus obtain the following variational equation:

$$\dot{\boldsymbol{\xi}} = [I_{M \times N} \otimes Df(\mathbf{s}) - c(\mathcal{L}^L \otimes H) - d(\mathcal{L}^I \otimes \Gamma)]\boldsymbol{\xi}, \quad (3)$$

where $\boldsymbol{\xi} = \mathbf{x} - \mathbf{1}_M \otimes \mathbf{1}_N \otimes \mathbf{s}$.

Suppose that \mathcal{L}^L and \mathcal{L}^I are symmetric matrices, and satisfy $\mathcal{L}^L \mathcal{L}^I = \mathcal{L}^I \mathcal{L}^L$. After diagonalization and decoupling (see the Methods section for details), we get the multiplex master stability equation for a system described by Eq. (1):

$$\dot{\mathbf{y}} = [Df(\mathbf{s}) - \alpha H - \beta \Gamma] \mathbf{y}, \quad (4)$$

where $\alpha = c\lambda$, $\beta = d\mu$, λ and μ are the eigenvalues of \mathcal{L}^L and \mathcal{L}^I respectively, and satisfy $\lambda^2 + \mu^2 \neq 0$.

Since this equation may be a time-varying system, particularly if $s(t)$ is a function of time, its eigenvalues may not be useful for determining the stability. Therefore, the largest Lyapunov exponent (LLE) of Eq. (4) is used instead, which is a function of α and β , denoted $LLE(\alpha, \beta)$ and called the multiplex Master Stability Function for Eq. (1).

When $\lambda \neq 0$ and $\mu = 0$, there is no inter-layer couplings regardless of d , for d arbitrarily chosen in $[0, +\infty)$, and Eq. (4) reduces to

$$\dot{\mathbf{y}} = [Df(\mathbf{s}) - \alpha H] \mathbf{y}, \quad (5)$$

It is clear that Eq. (5) becomes exactly the master stability equation of each independent intra-layer network (no inter-layer couplings).

Similarly, when $\lambda = 0$ and $\mu \neq 0$, we can obtain the following equation

$$\dot{\mathbf{y}} = [Df(\mathbf{s}) - \beta \Gamma] \mathbf{y}, \quad (6)$$

regardless of coupling strength c , for c arbitrarily chosen in $[0, +\infty)$. Eq. (6) becomes exactly the master stability equation for each independent inter-layer network (no intra-layer couplings).

For a single layer network, a necessary condition for the synchronization manifold to be stable is that the largest Lyapunov exponent $LLE(\alpha)$ of Eq. (5) less than zero [41]. In analogy to a single layer, for the multiplex master stability equation (4), $LLE(\alpha, \beta) < 0$ is a necessary condition for stability of the synchronization manifold in a multiplex network.

It is worth noting in particular the case when the intra- and inter-layer coupling functions are identical. Here $H = \Gamma$, and Eq. (4) turns into $\dot{\mathbf{y}} = [Df(\mathbf{s}) - \gamma H] \mathbf{y}$, (with $\gamma = \alpha + \beta$). This is the master stability equation for the corresponding single composite network where a single supra-Laplacian can describe its topology. That is to say, the master stability equation of the single composite network is a special case of Eq. (4).

Synchronized regions. Using the multiplex master stability equations developed above, we can analyze three types of synchronization behaviors: complete synchronization, intra-layer synchronization and inter-layer synchronization. The corresponding regions where these behaviors are stable are denoted respectively by SR , SR^{Intra} and SR^{Inter} . For the full multiplex network, from the multiplex master stability equation (4) with $\lambda \neq 0$ and $\mu \neq 0$, we can calculate the region

$$SR = \{(\alpha, \beta) | LLE(\alpha, \beta) < 0, \alpha \geq 0, \beta \geq 0\},$$

which is called the complete synchronized region. Whenever $LLE(\alpha, \beta) < 0$, perturbations transverse to the complete synchronization manifold die out, and the network is said to be synchronizable.

From Eq. (5), we can get the region for intra-layer synchronization ($\beta = 0$)

$$SR^{Intra} = \{(\alpha, d) | LLE(\alpha) < 0, d \geq 0\},$$

where $LLE(\alpha)$ is the largest Lyapunov exponent for master stability equation (5). Similarly, from Eq. (6), we obtain the region for inter-layer synchronization ($\alpha = 0$)

$$SR^{Inter} = \{(c, \beta) | LLE(\beta) < 0, c \geq 0\}$$

When network topological structures are given and fixed we can determine the eigenvalues λ and μ directly, and then the regions SR , SR^{Intra} and SR^{Inter} can be parameterized simply in terms of coupling strengths c and d , denoted by $SR_{c,d}$, $SR_{c,d}^{Intra}$ and $SR_{c,d}^{Inter}$. We call these regions the corresponding synchronized regions with respect to couplings c and d .

Two-layer network of Rössler oscillators. With the multiplex MSF framework developed, we now analyze in more depth a specific example of a two-layer network of Rössler oscillators and calculate the different types of synchronized regions.

The famous Rössler chaotic oscillator is described as

$$\begin{cases} \dot{x} = -y - z, \\ \dot{y} = x + ay, \\ \dot{z} = z(x - c) + b, \end{cases} \quad (7)$$

where $a = b = 0.2$ and $c = 9$. This is the function f in the multiplex network Eq. (1). That is, the state of each node in the network is a three-dimensional vector with each component evolving by Eq. (7). For inner coupling matrices H and Γ , we consider

the family of choices that fit the simplest form $I_{ij} \in \mathbb{R}^{3 \times 3}$ (where $i, j = 1, 2, 3$), which represents a matrix whose (i, j) -element is one and other elements are zero. The inter-layer topology is set to be one-to-one connection, that is to say, each node in one layer is connected to a counterpart node in the other layer.

Intra-layer and inter-layer synchronization. It is well known that complete synchronization means all the nodes in a network come to an identical state. But for multiplex networks, it is also very significant to study intra-layer synchronization and inter-layer synchronization. As shown in Fig. 1, intra-layer synchronization means all the nodes within each layer reach an identical state, while inter-layer synchronization means each node in a layer reaches the same state as its counterparts in other layers.

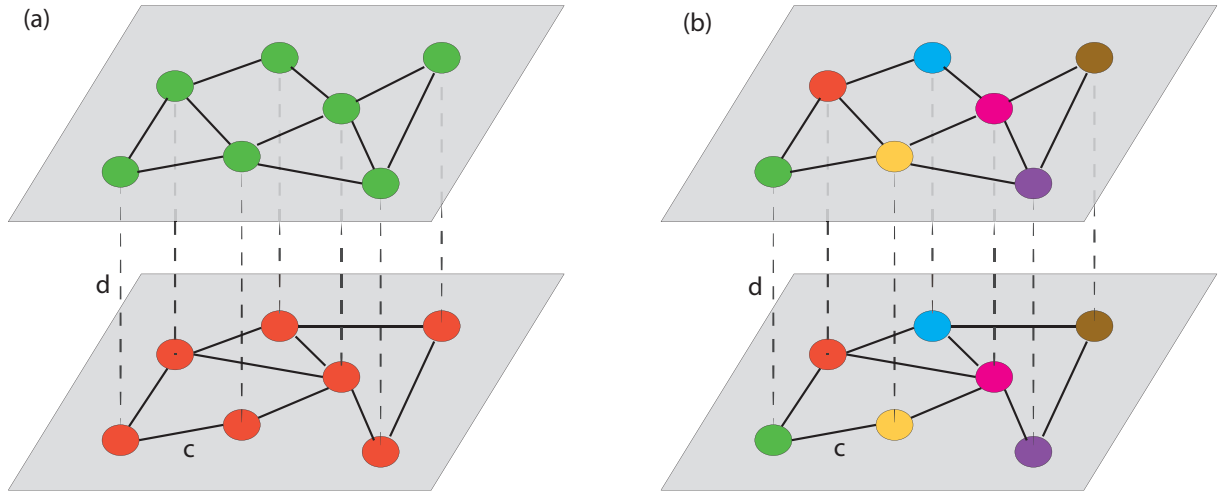


Figure 1: Schematic representation of (a) intra-layer synchronization and (b) inter-layer synchronization, in a multiplex network of two layers.

Complete synchronized regions for unknown intra-layer topologies. The regions of synchronization calculated from the multiplex MSF are parameterized by α and β , and thus do not require that the inter- and intra-layer topology are specified. Figure 2 shows the complete synchronized regions as parameterized by (α, β) for a two-layer multiplex network of Rössler oscillators with arbitrary topology for different combinations of inter-layer and intra-layer coupling matrices H and Γ .

Synchronization occurs in the region when the MSF criterion is negative, in other words when $LLE(\alpha, \beta) < 0$. Thus from Fig. 2, we can easily obtain the complete synchronized regions $SR = \{(\alpha, \beta) | 0.2 < \alpha + \beta < 4.6\}$ for $H = I_{11}$ and $\Gamma = I_{11}$, $SR \approx \{(\alpha, \beta) | 0.23 <$

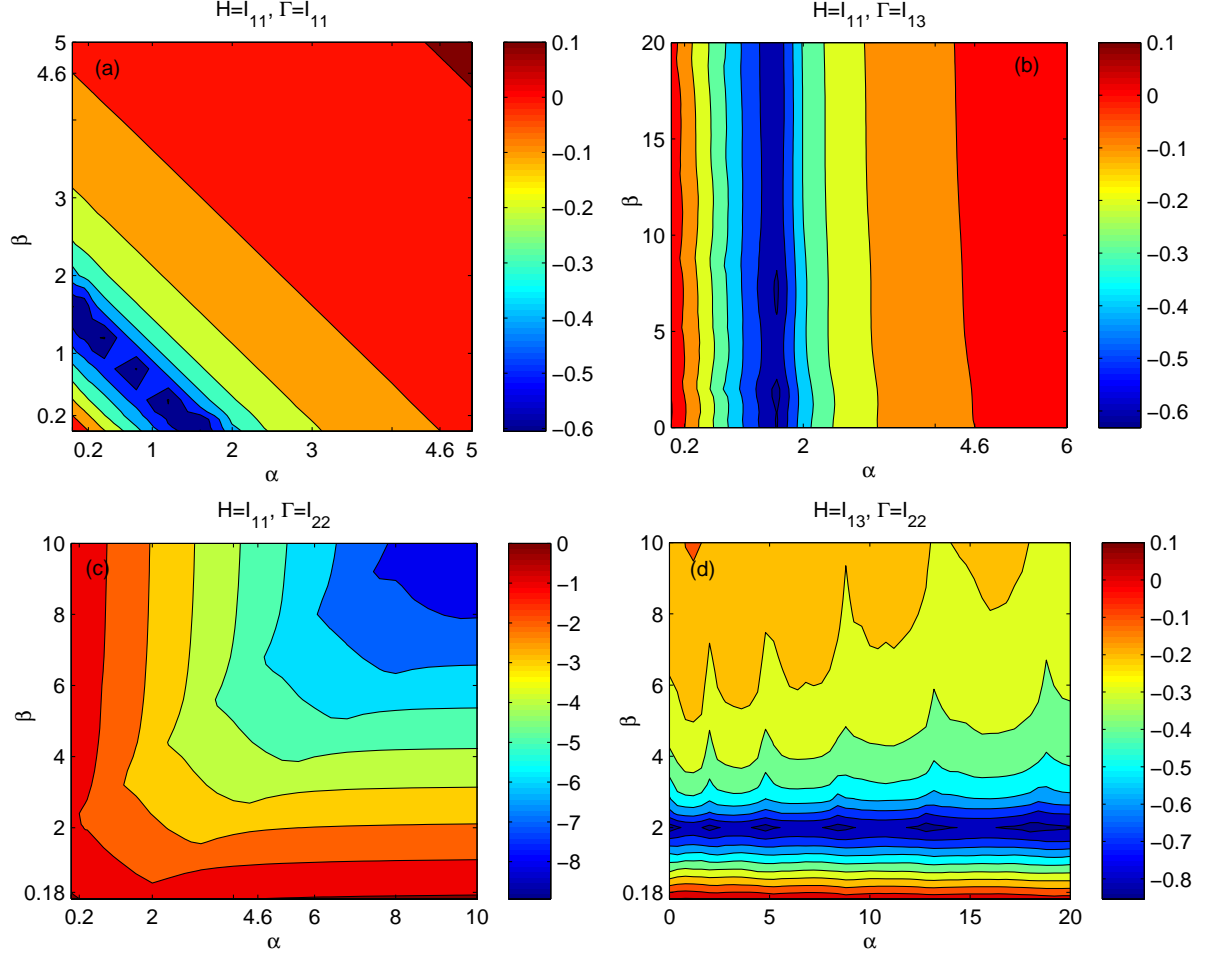


Figure 2: Shown is the value of $LLE(\alpha, \beta)$, the largest Lyapunov exponent of Eq. (4), as a function of parameters α and β . The complete synchronized region, SR , contains all the points with $LLE < 0$. Here the Rössler oscillator is taken as nodal dynamics, and the intra-layer coupling matrix H and the inter-layer coupling matrix Γ are chosen as follows: (a) $H = I_{11}$, $\Gamma = I_{11}$, (b) $H = I_{11}$, $\Gamma = I_{13}$, (c) $H = I_{11}$, $\Gamma = I_{22}$, (d) $H = I_{13}$, $\Gamma = I_{22}$.

$\alpha < 4.3, \beta \geq 0\}$ for $H = I_{11}$ and $\Gamma = I_{13}$, $SR \approx \{(\alpha, \beta) | \frac{\alpha}{0.2} + \frac{\beta}{0.18} > 1, \beta > -10^{-5}\alpha^4 + 0.00057\alpha^3 - 0.012\alpha^2 + 0.12\alpha - 0.35\}$ for $H = I_{11}$ and $\Gamma = I_{22}$, and $SR \approx \{(\alpha, \beta) | \beta > 0.2, \alpha \geq 0\}$ for $H = I_{13}$ and $\Gamma = I_{22}$.

Generally speaking, a multiplex network with a specified topology can achieve complete synchronization when all the nonzero network characteristic modes, including those of the intra-layer and inter-layer Laplacians, fall into the synchronized region. However, this is only a necessary condition. Even though all the nonzero characteristic modes fall into the region, the network might not get into complete synchronization but instead could support other coherent dynamical behaviors, such as intra-layer or inter-layer synchronization.

Synchronized regions with given intra-layer topologies. To push the analysis further, we must specify the topology of the two-layer Rössler oscillator network. For simplicity, assume that the two layers have the same intra-layer topology, and each node in one layer is connected with its replica in the other layer. Consider that each layer is a star network consisting of 5 nodes. Then, the intra-layer Laplacian matrix is

$$L = \begin{pmatrix} 5 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and the intra-layer supra-Laplacian matrix is $\mathcal{L}^L = \begin{pmatrix} L & 0 \\ 0 & L \end{pmatrix}$. The inter-layer Laplacian matrix $L^I = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$, and the inter-layer supra-Laplacian matrix $\mathcal{L}^I = L^I \otimes I_5$. It is easy to verify that $\mathcal{L}^L \mathcal{L}^I = \mathcal{L}^I \mathcal{L}^L$, and the characteristic values of \mathcal{L}^L and \mathcal{L}^I are $\lambda = 0, 0, 1, 1, 1, 1, 1, 1, 5, 5$ and $\mu = 0, 0, 0, 0, 0, 2, 2, 2, 2, 2$, respectively.

We can calculate the eigenvalues λ and μ directly and parameterize the synchronized regimes by the coupling strengths c and d (rather than the more general α and β) for all the different combinations of the inner coupling matrices H and Γ . (See the Methods section for more details on transforming SR to $SR_{c,d}$.) Consequently, for $H = I_{11}$ and $\Gamma = I_{11}$, the region with respect to parameters c and d is

$$SR_{c,d} = \{(c, d) | 0.2 < c + 2d, c + 0.4d < 0.92\}.$$

Similarly, for $H = I_{11}$ and $\Gamma = I_{13}$, then $SR_{c,d} \approx \{(c, d) | 0.23 < c < 0.86, d \geq 0\}$; for $H = I_{11}$ and $\Gamma = I_{13}$, then $SR_{c,d} \approx \{(c, d) | \frac{c}{0.2} + \frac{d}{0.09} > 1, d > \frac{1}{2}(-625 \cdot 10^{-5}c^4 + 0.07125c^3 - 0.2c^2 + 0.6c - 0.35)\}$; and for $H = I_{13}$ and $\Gamma = I_{22}$, then $SR_{c,d} \approx \{(c, d) | d > 0.1, c \geq 0\}$. These regions, $SR_{c,d}$, are shown in Figs. 3-6 by the solid lines in panels (c) for the different choices of H and Γ considered.

To test our theoretical predictions we next numerically solve the duplex Rössler networked system, and identify the parameter regions that support the three different coherent behaviors: complete synchronization, intra-layer synchronization and inter-layer synchronization. We quantify that the system has reached the specific type of behavior via the synchronization errors as defined in the Methods section. By bounding the values of these errors we develop three different indicator functions, which identify that the system has achieved macroscopic order of the form: $\text{Id}=3$ when the network reaches complete synchronization, $\text{Id}=2$ for intra-layer synchronization, $\text{Id}=1$ for inter-layer synchronization and $\text{Id}=0$ for none of the above cases. See Methods for full details.

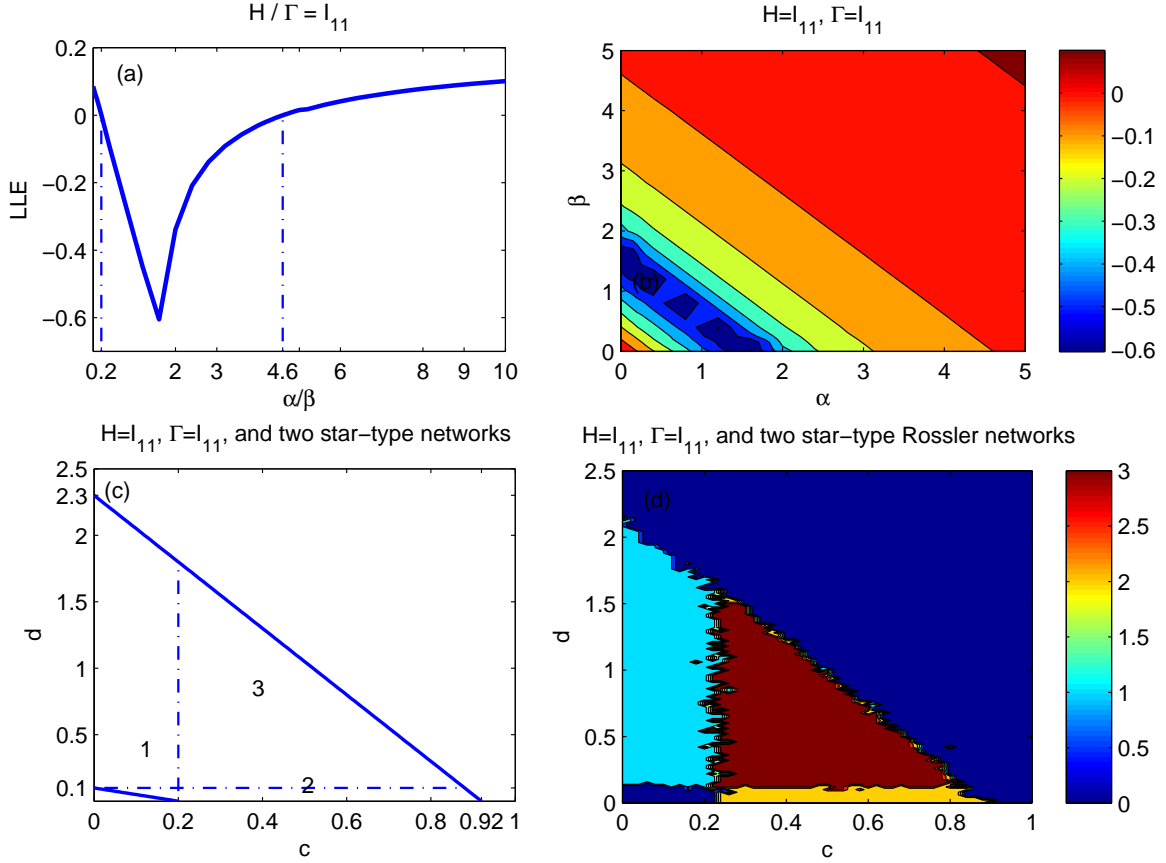


Figure 3: Network synchronized regions for $H = I_{11}$ and $\Gamma = I_{11}$. (a) the synchronized interval of the independent intra-layer/inter-layer Rössler network with respect to α/β ; (b) the synchronized region with respect to α and β for Rössler networks; (c) the synchronized region with respect to couplings c and d for a Rössler duplex consisting of two star layers with one-to-one inter-layer connections; (d) numerical synchronization areas with respect to couplings c and d , in which the maroon region represents complete synchronization area, the yellow is for intra-layer synchronization, and the cyan is inter-layer synchronization and the blue region represents non-synchronization.

Figure 3 shows network synchronized regions for the two-layer star network of Rössler oscillators for the scenario $H = I_{11}$ and $\Gamma = I_{11}$. In detail, panel (a) displays the synchronized intervals of the independent intra-layer and inter-layer Rössler networks with respect to α or β , which can be calculated from the master stability equations (5) and (6) (without consideration of d or c), respectively. Since $H = \Gamma$, the two intervals overlap. Panel (b) gives the synchronized region with respect to α and β for this Rössler network calculated from the master stability equation (4). Panel (c) shows the synchronized region as a function of intra- and inter-layer coupling strength c and d . Panel (d) shows the numerically calculated indicator function (i.e., the numerically calculated values of

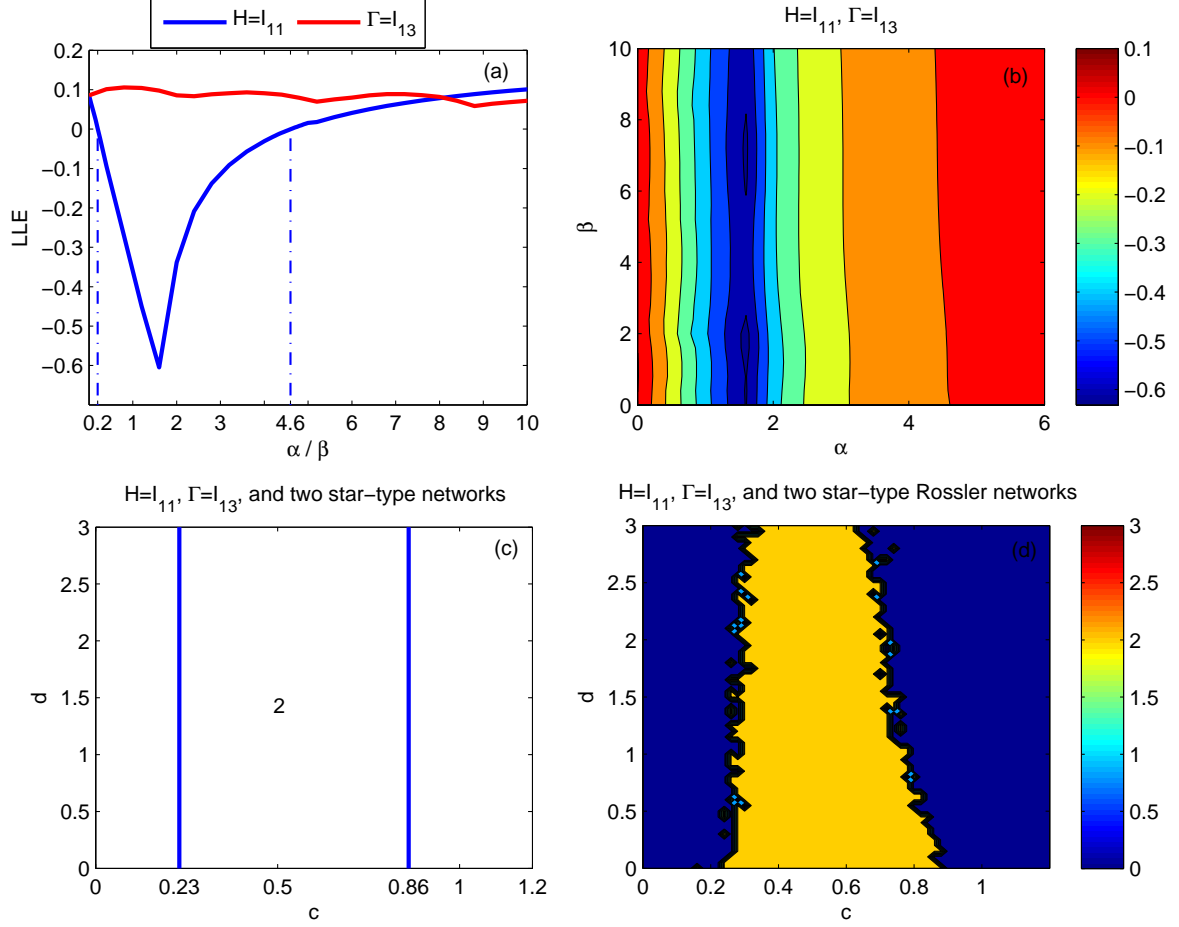


Figure 4: Network synchronized regions for $H = I_{11}$ and $\Gamma = I_{13}$. (a) the synchronized interval of the independent intra-layer/inter-layer Rössler network with respect to α/β ; (b) the synchronized region with respect to α and β for Rössler networks; (c) the synchronized region with respect to couplings c and d for a Rössler duplex consisting of two star layers with one-to-one inter-layer connections; (d) numerical synchronization areas with respect to couplings c and d , in which the maroon region represents complete synchronization area, the yellow is for intra-layer synchronization, and the cyan is inter-layer synchronization and the blue region represents non-synchronization.

synchronization error as classified in Eq. (13) given in the Methods section) with respect to couplings c and d for this duplex Rössler network. Here the maroon area represents complete synchronization, the yellow area is for intra-layer synchronization, the cyan is for inter-layer synchronization, and the blue region represents cases otherwise.

Other choices of for the coupling functions H and Γ are shown in Figures 4-6 for this same two-layer star network of Rössler oscillators. The results are analogous to those in Fig. 3. Note that in all these figures the region of complete synchronization predicted by the multiplex MSF using Eq. (4), shown as the solid lines in panels (c), does not

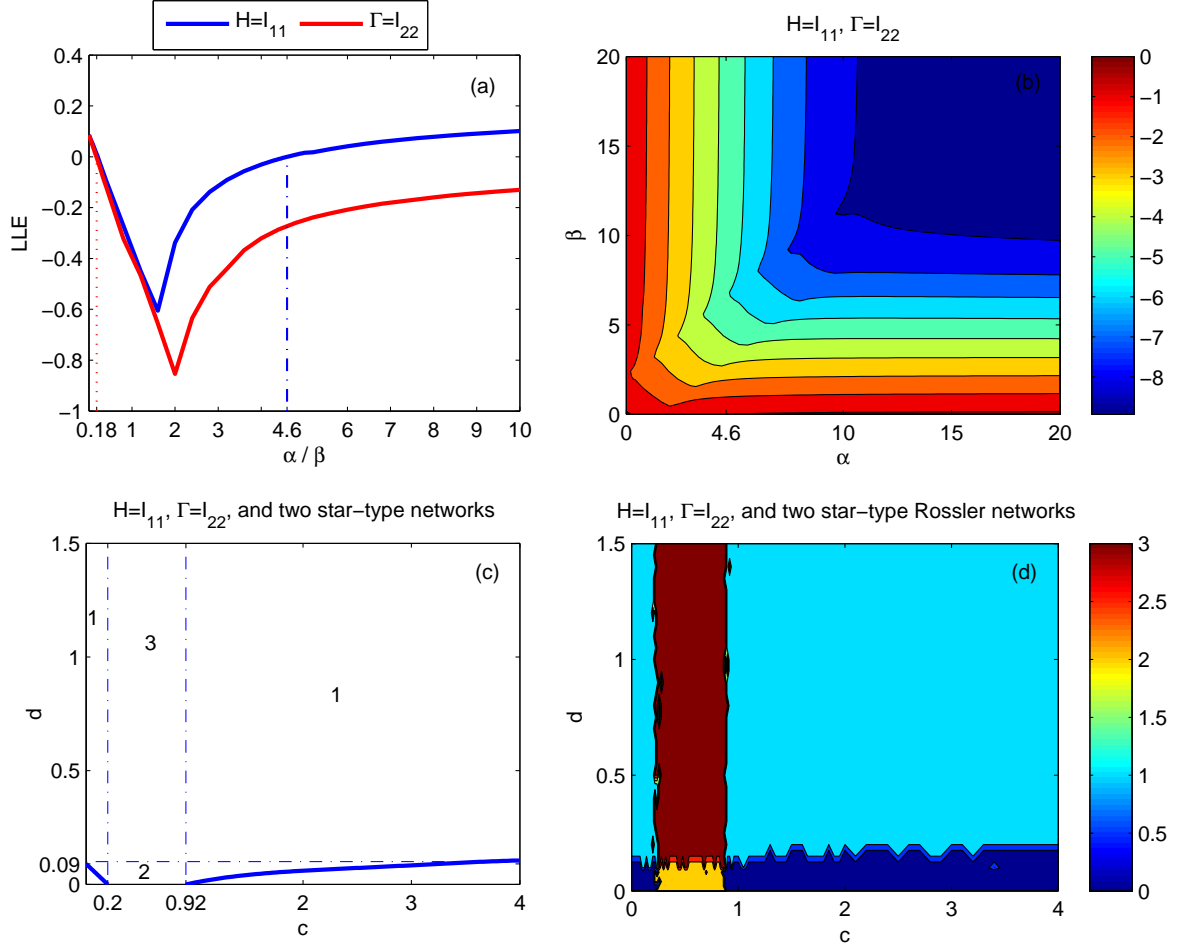


Figure 5: Network synchronized regions for $H = I_{11}$ and $\Gamma = I_{22}$. (a) the synchronized interval of the independent intra-layer/inter-layer Rössler network with respect to α/β ; (b) the synchronized region with respect to α and β for Rössler networks; (c) the synchronized region with respect to couplings c and d for a Rössler duplex consisting of two star layers with one-to-one inter-layer connections; (d) numerical synchronization areas with respect to couplings c and d , in which the maroon region represents complete synchronization area, the yellow is for intra-layer synchronization, and the cyan is inter-layer synchronization and the blue region represents non-synchronization.

capture all of the behaviors exhibited by direct numerical simulation shown in panels (d), which have areas of complete synchronization, intra-layer synchronization and inter-layer synchronization. Next we show how these distinct areas can be determined from the three regions: $SR_{c,d}$, $SR_{c,d}^{Intra}$, and $SR_{c,d}^{Inter}$ derived from Eqs. (4), (5), and (6). We will see that the characteristic modes falling into the synchronized region, $SR_{c,d}$ determined from Eq. (4), is a necessary but not sufficient condition for complete synchronization of the network. Instead the intersections of the regions determine the type of coherent behavior that is stable.

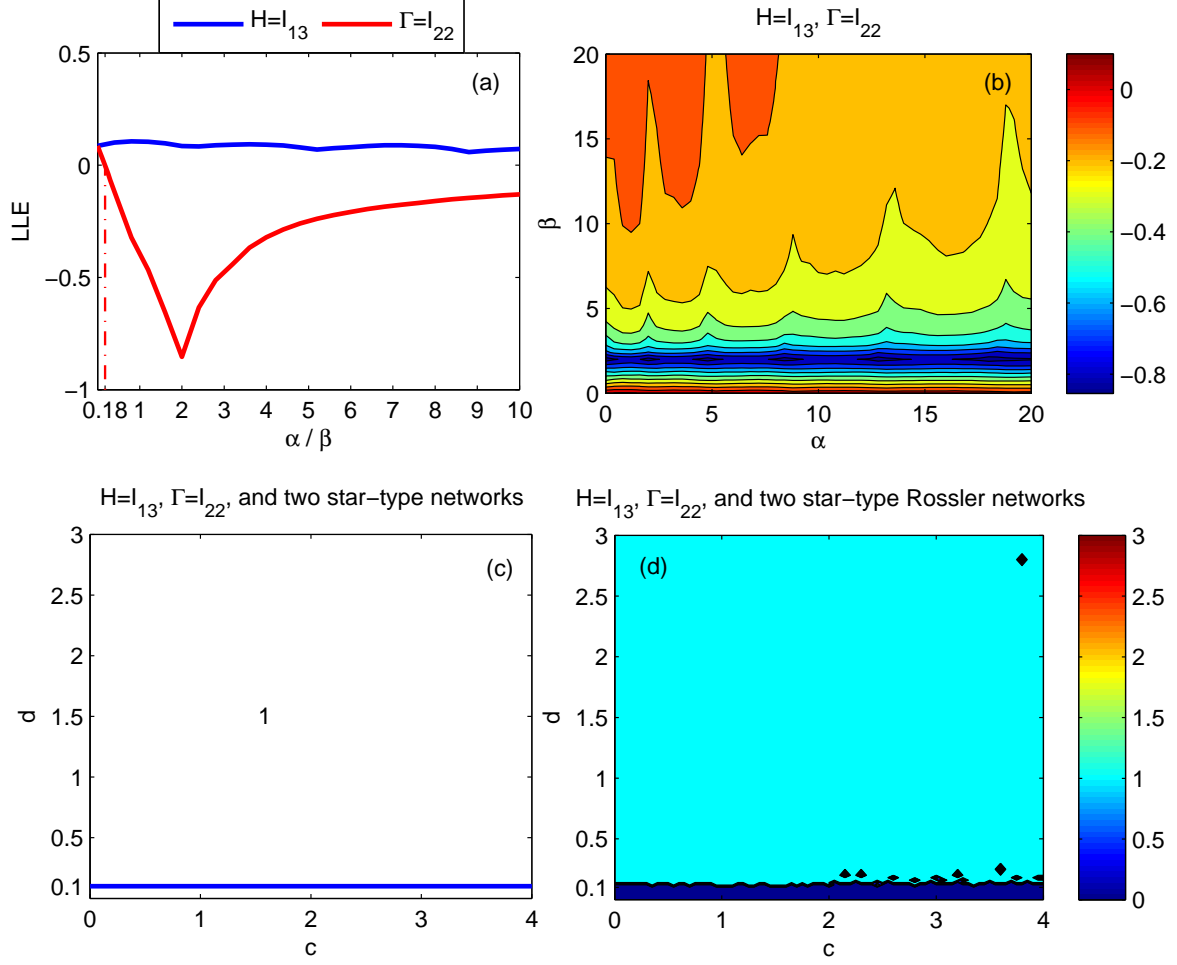


Figure 6: Network synchronized regions for $H = I_{13}$ and $\Gamma = I_{22}$. (a) the synchronized interval of the independent intra-layer/inter-layer Rössler network with respect to α/β ; (b) the synchronized region with respect to α and β for Rössler networks; (c) the synchronized region with respect to couplings c and d for a Rössler duplex consisting of two star layers with one-to-one inter-layer connections; (d) numerical synchronization areas with respect to couplings c and d , in which the maroon region represents complete synchronization area, the yellow is for intra-layer synchronization, and the cyan is inter-layer synchronization and the blue region represents non-synchronization.

For example, for the case with $H = I_{11}$ and $\Gamma = I_{11}$, the synchronized region $SR_{c,d} = \{(c,d) | c + 2d > 0.2, c + 0.4d < 0.92\}$, the intra-layer synchronized region $SR_{c,d}^{Intra} = \{(c,d) | 0.2 < c < 0.92, d \geq 0\}$ and the inter-layer synchronized region $SR_{c,d}^{Inter} = \{(c,d) | c \geq 0, 0.1 < d < 2.3\}$. The intersection of these three parts is $\{(c,d) | c > 0.2, d > 0.1, c + 0.4d < 0.92\}$, as labeled by number ‘3’ in panel (c) of Fig. 3, which essentially coincides with the numerically calculated complete synchronization area in maroon color in panel (d). Furthermore, the mere intra-layer synchronization (without inter-layer synchronization) area in yellow in panel (d) coincides with the region labeled as

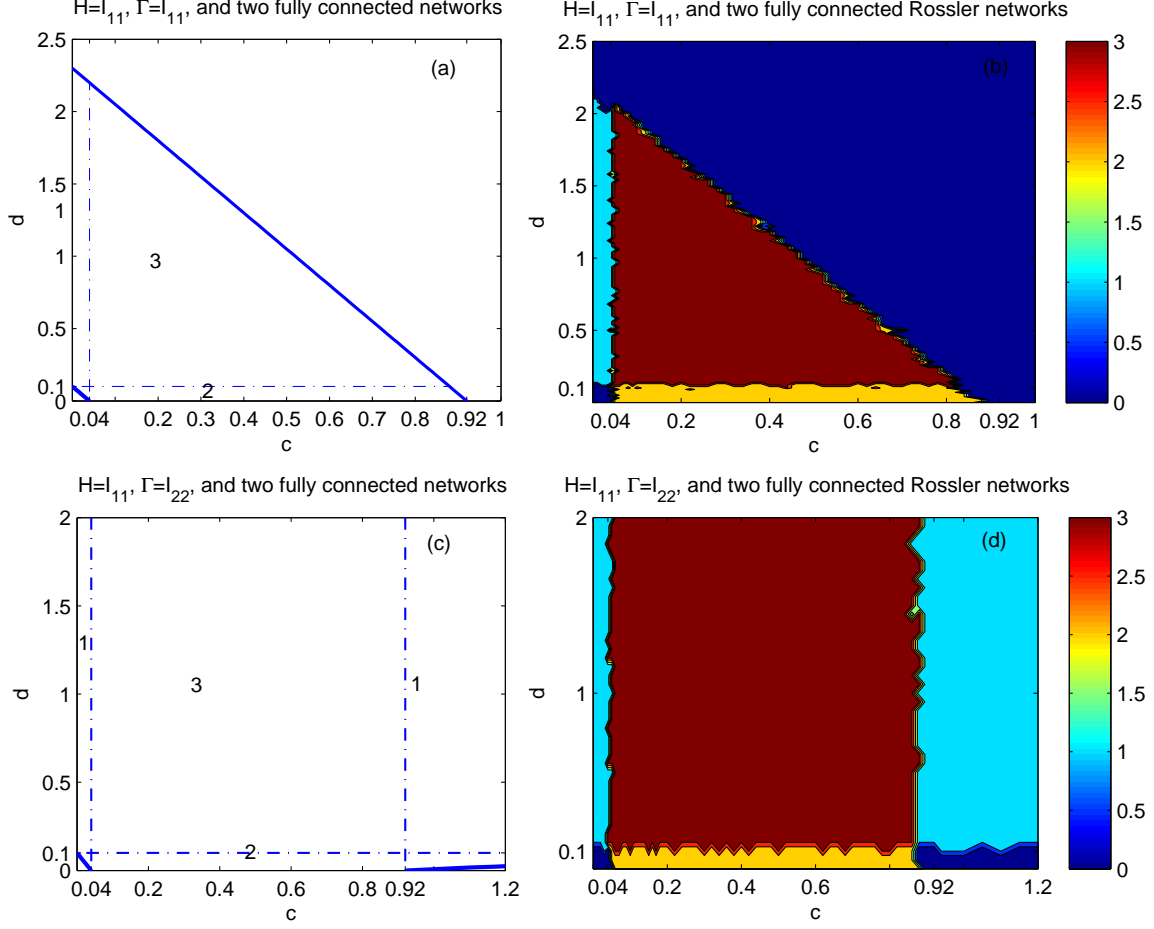


Figure 7: Network synchronized regions for Rössler networks composed of two single-layer fully connected networks with different H and Γ . (a) and (c) are the synchronized regions about c and d ; (b) and (d) are numerical synchronization areas, in which the maroon region means complete synchronization, the yellow region means mere intra-layer synchronization, the cyan region means mere inter-layer synchronization and the blue region means non-synchronization.

‘2’ in panel (c): $SR_{c,d} \cap SR_{c,d}^{Intra} - SR_{c,d}^{Inter} = \{(c,d) | c > 0.2, 0 \leq d < 0.1, c + 0.4d < 0.92\}$, and the mere inter-layer synchronization (without intra-layer synchronization) area in cyan agrees well with the region labeled as ‘1’ in panel (c): $SR_{c,d} \cap SR_{c,d}^{Inter} - SR_{c,d}^{Intra} = \{(c,d) | 0 \leq c < 0.2, d > 0.1, c + 0.4d < 0.92\}$. Similar observations can be obtained in panels (c) and (d) of Figs. 4-6.

In other words, the actual area for complete synchronization is determined by the intersection of $SR_{c,d}$, $SR_{c,d}^{Intra}$ and $SR_{c,d}^{Inter}$, that is, $SR_{c,d} \cap SR_{c,d}^{Intra} \cap SR_{c,d}^{Inter}$. Moreover, the mere intra-layer synchronization area is determined by the intersection of synchronized region and intra-layer synchronized region subtracting the inter-layer synchronized part, that is, $SR_{c,d} \cap SR_{c,d}^{Intra} - SR_{c,d}^{Inter}$. The mere inter-layer synchronization area is determined

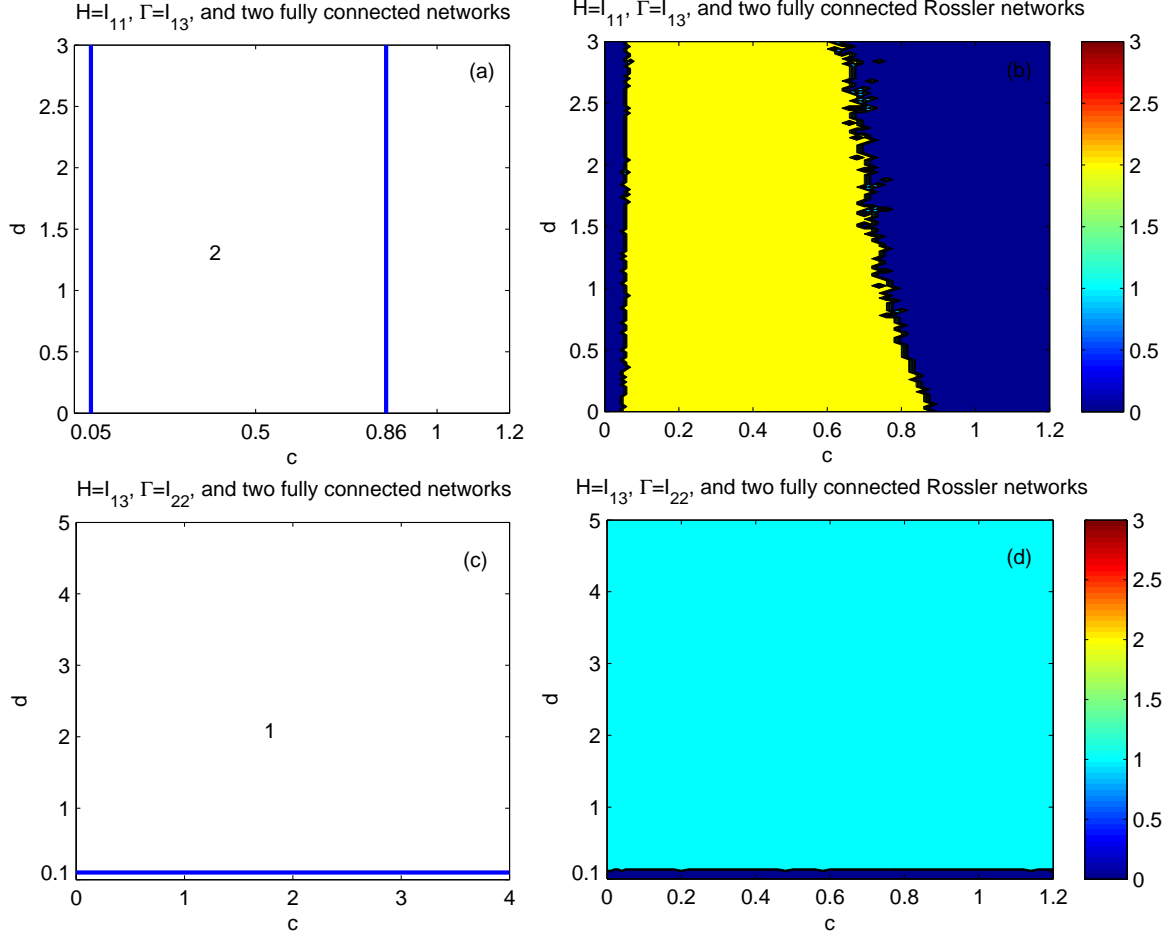


Figure 8: Network synchronized regions for Rössler networks composed of two single-layer fully connected networks with different H and Γ . (a) and (c) are the synchronized regions about c and d ; (b) and (d) are numerical synchronization areas, in which the maroon region means mere complete synchronization, the yellow region means mere intra-layer synchronization, the cyan region means inter-layer synchronization and the blue region means non-synchronization.

by $SR_{c,d} \cap SR_{c,d}^{Inter} - SR_{c,d}^{Intra}$.

Furthermore, when nodal dynamics and network structures are given, $SR_{c,d}^{Intra}$ and $SR_{c,d}^{Inter}$ are mainly determined by the inner coupling matrices of the intra-layer nodes (H) and the inter-layer nodes (Γ) respectively, and $SR_{c,d}$ is determined by both. Particularly, if the inter-layer coupling matrix Γ makes the inter-layer synchronized region $SR_{c,d}^{Inter}$ empty, then the multiplex network cannot achieve inter-layer synchronization, resulting in the failure of complete synchronization, as shown in Fig. 4. If the intra-layer coupling matrix H makes the intra-layer synchronized region $SR_{c,d}^{Intra}$ empty, then the multiplex network cannot achieve intra-layer synchronization, which also leads to failure of complete synchronization, as shown in Fig. 6.

In order to verify the previous results on a different multiplex topology, we consider a duplex network composed of two fully connected network layers with one-to-one inter-layer connections. The results shown in Figs. 7 and 8 again illustrate the above observations.

Discussion

In summary, we develop a master stability function framework which captures an essential feature of multiplex networks, that the intra-layer and inter-layer coupling functions can be distinct. Here we consider that edges are undirected and define a distinct supra-Laplacian matrix for intra-layer connections, denoted \mathcal{L}^L , and one for inter-layer connections, denoted \mathcal{L}^I . If \mathcal{L}^L and \mathcal{L}^I commute, the multiplex network can be decoupled and thus the characteristic modes of the intra-layer Laplacian are separated from those of the inter-layer one. We can then develop a multiplex master stability equation, Eq. (4), to establish the necessary region for complete synchronization. In the limit of no inter-layer coupling the multiplex MSF reduces to a master stability equation for each independent layer, Eq. (5), allowing us to calculate the necessary region for intra-layer synchronization. In the limit of no intra-layer coupling the multiplex MSF reduces to a master stability equation for each independent inter-layer network, Eq. (6), allowing us to calculate the necessary region for inter-layer synchronization.

To explicitly use the multiplex MSF framework requires specifying $f(\cdot)$ (i.e., the internal nodal dynamics), and the inter- and intra-layer coupling functions (i.e., H and Γ respectively). We consider specifically a two-layer network of Rössler oscillators and various forms of H and Γ . We find that the different types of coherent behaviors observed in the network are determined by the intersections of the three necessary regions describing complete synchronization, intra-layer synchronization and inter-layer synchronization. Given a specified network topology, these regions can then be parameterized by the intra- and inter-layer coupling strengths (i.e., c and d respectively). Complete synchronization is stable when both c and d fall into the overlap of the three regions. Intra-layer synchronization is stable when both c and d fall into the overlap of the complete synchronized region and the intra-layer synchronized region. Inter-layer synchronization is stable when both c and d fall into the overlap of the complete synchronized region and the inter-layer synchronized region.

For a given network nodal dynamics, the complete synchronized region is mainly determined by both inner coupling matrices H and Γ . Similarly, the intra-layer synchronized region is mainly determined by the intra-layer coupling matrix H , and the inter-layer

synchronized region by the inter-layer coupling matrix Γ . Therefore, in addition to nodal dynamics, the inner coupling function is an essential factor to determine which kind of synchronization the network will arrive at. If H is in such a form that the intra-layer synchronized region is empty, intra-layer synchronization is unstable regardless of however large the intra-layer coupling strength is. Similarly, if Γ is in such a form that the inter-layer synchronized region is empty, inter-layer synchronization is unstable regardless of however large the inter-layer coupling strength is. In either case, complete synchronization will not occur regardless of the coupling strength.

Here we have numerically investigated specific duplex networks of Rössler oscillators where the two layers have the same topological structure. Our approach can be applied to multiplex networks with different choices for the internal nodal dynamics and different network topologies on different layers. But it does require that the supra-Laplacian matrix of the intra-layer connections is commutative with that of the inter-layer connections.

Methods

Decoupling the multiplex network system. Suppose that supra-Laplacian matrices \mathcal{L}^L and \mathcal{L}^I are symmetric matrices, and satisfy $\mathcal{L}^L \mathcal{L}^I = \mathcal{L}^I \mathcal{L}^L$, then there exists an invertible matrix P such that

$$P^{-1} \mathcal{L}^L P = \text{diag}\{\lambda_1, \dots, \lambda_M, \lambda_{M+1}, \dots, \lambda_{M \times N}\},$$

$$P^{-1} \mathcal{L}^I P = \text{diag}\{\mu_1, \dots, \mu_M, \mu_{M+1}, \dots, \mu_{M \times N}\},$$

where $0 = \lambda_1 = \dots = \lambda_M < \lambda_{M+1} \leq \dots \leq \lambda_{M \times N}$, $\mu_k \geq 0$ ($k = 1, 2, \dots, M \times N$), and $\text{diag}\{v_1, \dots, v_M\}$ denotes a diagonal matrix whose j -th diagonal element is v_j ($j = 1, 2, \dots, M$).

By denoting a new vector $\boldsymbol{\eta} = [\boldsymbol{\eta}_1^\top, \boldsymbol{\eta}_2^\top, \dots, \boldsymbol{\eta}_{M \times N}^\top]^\top = (P \otimes I_m)^{-1} \boldsymbol{\xi}$, we can turn the variational equation (3) into

$$\dot{\boldsymbol{\eta}} = [I_{M \times N} \otimes Df(\mathbf{s}) - c(\text{diag}\{\lambda_1, \dots, \lambda_{M \times N}\} \otimes H) - d(\text{diag}\{\mu_1, \dots, \mu_{M \times N}\} \otimes \Gamma)] \boldsymbol{\eta}. \quad (8)$$

It further yields

$$\dot{\boldsymbol{\eta}}_k = [Df(\mathbf{s}) - c\lambda_k H - d\mu_k \Gamma] \boldsymbol{\eta}_k, \quad k = 1, 2, \dots, M \times N. \quad (9)$$

Here, $\boldsymbol{\eta}_k$ represents the mode of perturbation in the generalized eigenspace associated with λ_k and μ_k . A criterion for the synchronization manifold to be (asymptotically)

stable is that all the transversal Lyapunov exponents of the variational equation (9) are strictly negative. Clearly, these Lyapunov exponents depend on the node dynamics $f(\cdot)$, the network intra- and inter-layer coupling strengths c and d , and the coupling matrices H and Γ . Consequently, we can get the three master stability equations: Eqs. (4), (5) and (6).

Calculating synchronized regions $SR_{c,d}$. We can calculate three synchronized regions with regard to parameters α and β : SR , SR^{Intra} and SR^{Inter} from Eqs. (4), (5) and (6), respectively. Furthermore, when the network topologies are given, we can directly calculate the characteristic values of supra-Laplacian matrices and parameterize those regions in terms of c and d , since $\alpha = c\lambda$ and $\beta = d\mu$.

For example, when $H = I_{11}$ and $\Gamma = I_{11}$, the nonzero characteristic modes $\alpha = c\lambda$ and $\beta = c\mu$ should lie in $SR = \{(\alpha, \beta) | 0.2 < \alpha + \beta < 4.6\}$, and consequently the region with respect to parameters c and d is

$$SR_{c,d} = \{(c, d) | 0.2 < c + 2d, c + 0.4d < 0.92\}.$$

For other combinations of H and Γ , the synchronized regions with respect to parameters c and d can be similarly obtained.

Synchronization errors & Indicator function. To measure the extent of intra-layer, inter-layer and complete synchronization, we introduce the following indices:

$$E_{Intra}^{(k)}(t) = \frac{1}{N} \sum_{i=1}^N \|x_i^{(k)}(t) - \bar{x}^{(k)}(t)\|, \quad k = 1, 2, \dots, M \quad (10)$$

where $\|\cdot\|$ is a norm operator, and $\bar{x}^{(k)}(t)$ is the average state of all the nodes in the k th layer at time t . Thus $E_{Intra}^{(k)}(t)$ is the synchronization error of nodes in the k th layer at time t , namely, the intra-layer synchronization error.

Similarly, the inter-layer synchronization error is defined as

$$E_{Inter}(t) = \frac{1}{MN} \sum_{i=1}^N \sum_{k=1}^M \|x_i^{(k)}(t) - \bar{x}_i(t)\|, \quad (11)$$

and the complete synchronization error is defined as

$$E(t) = \frac{1}{NM} \sum_{k=1}^M \sum_{i=1}^N \|x_i^{(k)}(t) - \bar{x}(t)\|, \quad (12)$$

where $\bar{x}_i(t)$ is the average state of the node i in each layer and its counterparts in other layers, and $\bar{x}(t)$ is that of all the nodes in the multiplex network.

With these definitions, we use the following indicator function to represent complete synchronization, intra-layer synchronization and inter-layer synchronization:

$$Id = \begin{cases} 3, & E_{Inter}(t) < \epsilon \text{ and } E_{Intra}^{(k)}(t) < \epsilon \text{ for all } t > T_0, \\ 2, & E_{Inter}(t) \geq \epsilon \text{ and } E_{Intra}^{(k)}(t) < \epsilon \text{ for all } t > T_0, \\ 1, & E_{Inter}(t) < \epsilon \text{ and } E_{Intra}^{(k)}(t) \geq \epsilon \text{ for all } t > T_0, \\ 0, & \text{other.} \end{cases} \quad (13)$$

Here, T_0 is a time threshold value and ϵ is a given threshold for synchronization errors. In the simulations, $\epsilon = 1.0 \times 10^{-2}$, and $T_0 = 0.8T_{total}$ (T_{total} is the total evolution time). It is obvious that the network reaches complete synchronization when $Id = 3$, intra-layer synchronization when $Id = 2$, inter-layer synchronization when $Id = 1$, and none of the above when $Id = 0$.

Acknowledgments

This work is supported in part by the National Key Research and Development Program of China under Grant 2016YFB0800401, in part by the National Natural Science Foundation of China under Grants 61573004, 11501221, 61573262, 61621003 and 61532020, in part by the U.S. Army Research Laboratory and the U.S. Army Research Office under Multidisciplinary University Research Initiative Award No. W911NF-13-1-0340, in part by the Promotion Program for Young and Middle-aged Teacher in Science and Technology Research of Huaqiao University (ZQN-YX301), and in part by the Natural Science Foundation of Fujian Province (2015J01260).

References

- [1] Watts, D. J. & Strogatz, S. H. Collective dynamics of ‘small-world’ networks. *Nature* **393**, 440–442 (1998).
- [2] Barahona, M. & Pecora, L. M. Synchronization in small-world systems. *Phys. Rev. Lett.* **89**, 054101 (2002).
- [3] Hong, H., Choi, M. Y. & Kim, B. J. Synchronization on small-world networks. *Phys. Rev. E* **65**, 026139 (2002).
- [4] Newman, M. E. J. The Structure and Function of Complex Networks. *SIAM Rev.* **45**, 167–256 (2003).

- [5] Lü, J., Yu, X., Chen, G. & Cheng, D. Characterizing the synchronizability of small-world dynamical networks. *IEEE Trans. Circuits Syst. I: Regular Papers* **51**, 787–796 (2004).
- [6] Lü, J. & Chen, G. A time-varying complex dynamical network model and its controlled synchronization criteria. *IEEE Trans. Automat. Contr.* **50**, 841–846 (2005).
- [7] Zhou, J., Lu, J. & Lü, J. Adaptive synchronization of an uncertain complex dynamical network. *IEEE Trans. Automat. Contr.* **51**, 652–656 (2006).
- [8] Boccaletti, S., Latora, V., Moreno, Y., Chavez, M. & Hwang, D. U. Complex networks: Structure and dynamics. *Phys. Rep.* **424**, 175–308 (2006).
- [9] Wu, C. W. *Synchronization in Complex Network of Nonlinear Dynamical System*. pp. 51–123 (World Scientific, Singapore, 2007).
- [10] Arenas, A., Díaz-Guilera, A., Kurths, J., Moreno, Y. & Zhou, C. Synchronization in complex networks. *Phys. Rep.* **469**, 93–153 (2008).
- [11] Chen, Y., Lü, J., Han, F. & Yu, X. On the cluster consensus of discrete-time multi-agent systems. *Syst. Contr. Lett.* **60**, 517–523 (2011).
- [12] Huang, L., Lai, Y. C. & Gatenby, R. A. Optimization of synchronization in complex clustered networks. *Chaos* **18**, 013101 (2008).
- [13] Donetti, L., Hurtado, P. I. & Muñoz, M. A. Entangled networks, synchronization, and optimal network topology. *Phys. Rev. Lett.* **95**, 188701 (2005).
- [14] Pecora, L. M., Sorrentino, F., Hagerstrom, A. M., Murphy, T. E. & Roy, R. Cluster synchronization and isolated desynchronization in complex networks with symmetries. *Nat. Commun.* **5**, 4079, (2014).
- [15] Tang, L., Lu, J.A. & Chen, G. Synchronizability of small-world networks generated from ring networks with equal-distance edge additions. *Chaos* **22**, 023121 (2012).
- [16] D’Agostino, G. & Scala, A. *Networks of Networks: the last Frontier of Complexity* (Springer, Berlin, 2014).
- [17] Brummitt, C. D., D’Souza, R. M. & Leicht, E. A. Suppressing cascades of load in interdependent networks, *Proc. Natl Acad. Sci. U.S.A.* **109**, E680–E689 (2012).
- [18] Gómez, S., Díaz-Guilera, A., Gómez-Gardeñes, J., Pérez-Vicente, C. J., Moreno, Y. & Arenas, A. Diffusion dynamics on multiplex networks. *Phys. Rev. Lett.* **110**, 028701 (2013).

- [19] Radicchi, F., & Arenas, A. Abrupt Transition in the Structural Formation of Interconnected Networks. *Nat. Phys.* **9**, 717 (2013).
- [20] De Domenico, M., Solé-Ribalta, A., Gómez, S. , & Arenas, A. Navigability of Interconnected Networks under Random Failures. *Proc. Natl. Acad. Sci. U.S.A.* **111**, 8351 (2014).
- [21] Valles-Catala, T., Massucci, F. A., Guimera, R., & Sales-Pardo, M. Multilayer stochastic block models reveal the multilayer structure of complex networks. *Phys. Rev. X* **6**, 011036 (2016).
- [22] Kivelä, M., Arenas, A., Barthélemy, M., Gleeson, J.P., Moreno, Y., & Porter, M.A. Multilayer networks. *J. Complex Netw.* **2** (3) 203-271 (2014).
- [23] Boccaletti, S., Bianconi, G., Criado, R., Del Genio, C.I., Gómez-Gardeñes, J., Romance, M., Sendiña-Nadal, I., Wang, Z. and Zanin, M., The structure and dynamics of multilayer networks. *Phys. Rep.*, **544**(1), 1-122 (2014).
- [24] Solé-Ribalta, A., Domenico, M. D., Kouvaris, N. E., Díaz-Guilera, A., Gómez, S. & Arenas, A. Spectral properties of the Laplacian of multiplex network. *Phys. Rev. E* **88**, 032807 (2013).
- [25] Aguirre, J., Sevilla-Escoboza, R., Gutiérrez, R., Papo, D. & Buldú, J. M. Synchronization of interconnected networks: the role of connector nodes. *Phys. Rev. Lett.* **112**, 248701 (2014).
- [26] Boccaletti, S., Bianconi, G., Criado, R., Del Genio, C. I., Gómez-Gardeñes, J., Romance, M., Sendiña-Nadal, I., Wang, Z. & Zanin, M. The structure and dynamics of multilayer networks. *Phys. Rep.* **544**, 1–122 (2014).
- [27] De Domenico, M., Nicosia, V., Arenas, A. & Latora, V. Structural reducibility of multilayer networks. *Nat. Commun.* **6**, 6864 (2015).
- [28] Lu, R., Yu, W., Lü, J. & Xue A. Synchronization on complex networks of networks. *IEEE Trans. Neural Netw. Learn. Syst.* **25**, 2110–2118 (2014).
- [29] Luo, C., Wang, X. & Liu, H. Controllability of time-delayed Boolean multiplex control networks under asynchronous stochastic update. *Sci. Rep.* **4**, 7522 (2014).
- [30] Xu, M., Zhou, J., Lu, J. A. & Wu, X., Synchronizability of two-layer networks. *Eur. Phys. J. B* **88**, 240 (2015).
- [31] Li, Y., Wu, X., Lu, J. A. & Lü, J. Synchronizability of duplex networks. *IEEE Trans. Circuits Sys. II: Express Briefs* **63**, 206–210 (2015).

- [32] Gambuzza, L.V., Frasca, M. & Gómez-Gardeñes, J. Intra-layer synchronization in multiplex networks, *EPL* **110**, 20010 (2015).
- [33] Sevilla-Escoboza, R., Sendiña-Nadal, I., Leyva, I., Gutiérrez, R., Buldú, J. M. & Boccaletti, S. Inter-layer synchronization in multiplex networks of identical layers. *Chaos* **26**, 065304 (2016).
- [34] Pecora, L. M. & Carroll, T. L. Master stability functions for synchronized coupled systems. *Phys. Rev. Lett.* **80**, 2109–2112 (1998).
- [35] Tang, L., Lu, J. A., Lü, J. & Yu, X. Bifurcation analysis of synchronized region in complex dynamical network. *Int. J. Bifurcat. Chaos* **22**, 1250282 (2012).
- [36] Tang, L., Lu, J. A., Lü, J. & Wu, X. Bifurcation analysis of synchronized region in complex dynamical networks with coupling delay. *Int. J. Bifurcat. Chaos* **24**, 145001 (2014).
- [37] Tang, L., Wu, X., Lü, J. & Lu, J. A. Bifurcation behaviors of synchronized regions in Logistic map networks with coupling delay. *Chaos* **25**, 033101 (2015).
- [38] Sorrentino, F. Synchronization of hypernetworks of coupled dynamical systems, *New J. Phys.* **14**, 033035 (2012).
- [39] Irving D. & Sorrentino, F. Synchronization of dynamical hypernetworks: Dimensionality reduction through simultaneous block-diagonalization of matrices, *Phys. Rev. E* **86**, 056102 (2012).
- [40] Charo I. d.G, Gómez-Gardeñes, I. Bonamassa, S. Boccaletti. Synchronization in networks with multiple interaction layers. arXiv:1611.05406.
- [41] Leonov, G. A. & Kuznetsov, N. V. Time-varying linearization and the Perron effects. *Int. J. Bifurcat. Chaos* **17**, 1079–1107 (2007).